A Motivic Snaith Decomposition

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Abstract The filtration $\mathrm{BGL}_0 \subset \ldots \subset \mathrm{BGL}_{n-1} \subset \mathrm{BGL}_n$ is split by motivic Becker–Gottlieb transfers in the motivic stable homotopy category over any scheme. This recovers results by Snaith on the splitting of $\mathrm{BGL}_n(\mathbb{C})$ in classical stable homotopy theory by passing to complex realizations. On the way, we extend motivic homotopy theory to smooth ind-schemes as bases and show how to construct the necessary fragment of the six operations and duality for this extension.

Contents

1. Introduction

Snaith [Sna79; Sna78] described splittings of the natural filtrations on $\mathrm{BGL}_n(\mathbb{C})$ and $\mathrm{BGL}_n(\mathbb{H})$ in the classical stable homotopy category, using Becker–Gottlieb transfer associated with the normalizer of a maximal torus in GL_n. He also obtained coarser results for $\mathrm{BGL}_n(\mathbb{R})$ and for finite fields. Using transfers associated with the block-diagonal subgroup $\mathrm{GL}_i \times \mathrm{GL}_{n-i} \subset \mathrm{GL}_n$, Mitchell and Priddy [MP89] recovered Snaith's results on $\mathrm{BGL}_n(\mathbb{C})$ and $\mathrm{BGL}_n(\mathbb{H})$ and improve the ones on $\mathrm{BGL}_n(\mathbb{R})$ and $\mathrm{BGL}_n(\mathbb{F}_q)$. In fact, the applicability of this approach to the case of $\mathrm{BGL}_n(\mathbb{C})$ was first noted by Richter.

In this paper, we prove an analogue of Snaith's, Mitchell and Priddy's and Richter's result in the motivic stable homotopy category of an arbitrary base scheme S:

Theorem 1.1. *Over any scheme S, there is a $\mathbb{P}^1_S$–stable splitting* $\mathrm{BGL}_{m,+} \simeq \bigvee_{i=1}^{m} \mathrm{BGL}_i/\mathrm{BGL}_{i-1}$ *of the natural filtration* $\mathrm{BGL}_0 \subset \ldots \subset \mathrm{BGL}_{m-1} \subset \mathrm{BGL}_m$.

Using topological realization, for $S = \mathrm{Spec}(\mathbb{C})$ or $S = \mathrm{Spec}(\mathbb{R})$, this recovers the classical story. Our approach mirrors the technique of Mitchell and Priddy [MP89], utilizing Becker–Gottlieb transfers associated with the block-diagonal inclusion $\mathrm{BGL}_i \times \mathrm{BGL}_{n-i} \longrightarrow \mathrm{BGL}_n$. We make essential use of the six operations approach to the motivic Becker–Gottlieb transfer introduced by Hoyois [Hoy14]. The main technical difficulty in our approach to the splitting of $\mathrm{BGL}_{n,+}$ is the fact that BGL_n is not representable by a scheme. However, it is representable by a smooth ind-scheme.

Overview To obtain a transfer for the inclusion $\mathrm{BGL}_i \times \mathrm{BGL}_{n-i} \longrightarrow \mathrm{BGL}_n$ we use an explicit model of BGL_n as the infinite Grassmannian Gr_n. We give an explicit presentation of the inclusion $\mathrm{BGL}_i \times \mathrm{BGL}_{n-i} \longrightarrow \mathrm{BGL}_n$ as a smooth Zariski–locally trivial bundle $U \longrightarrow \mathrm{Gr}_n$ over the ind-scheme Gr_n. Then we construct a symmetric monoidal presentable stable ∞–category $\mathcal{SH}(\mathrm{Gr}_n)$ which accepts a functor from smooth ind-schemes over Gr_n and supports a functor $\mathcal{SH}(\mathrm{Gr}_n) \longrightarrow \mathcal{SH}(S)$. The motivic Becker–Gottlieb transfer of $U \longrightarrow \mathrm{Gr}_n$ is then defined to be the image in $\mathcal{SH}(S)$ of the monoidal trace of U in $\mathcal{SH}(\mathrm{Gr}_n)$. We appeal to a result of May [May01] to see that this construction of Becker–Gottlieb transfers has a Mayer–Vietoris property. This enables us to show that, just like in classical topology, the transfers of the inclusions $\mathrm{BGL}_i \times \mathrm{BGL}_{n-i} \subset \mathrm{BGL}_n$ can be used to split the natural filtration on BGL_n. Since the transfer is a stable phenomenon, this splitting is naturally only found in the stable motivic world.

Notation In what follows S will be an arbitrary base scheme. For a scheme X we write $\mathcal{H}(X)$ for the ∞–category of presheaves of spaces on Sm_X localized at Nisnevich-local equivalences and projections $Y \times \mathbb{A}^1 \longrightarrow Y$. It is a presentable ∞–category in the sense of [Lur09]. We refer to $\mathcal{H}(X)$ interchangably as the $\mathbb{A}^1$–homotopy category of X or the motivic homotopy category of X. The associated pointed ∞–category will be denoted by $\mathcal{H}_\bullet(X)$. Inverting $(\mathbb{P}^1, \infty) \in \mathcal{H}_\bullet(X)$ with respect to the smash product yields the stable motivic homotopy category $\mathcal{SH}(X)$ of X. It is a symmetric monoidal, presentable, stable ∞–category in the sense of [Lur12]. An account of this definition of $\mathcal{H}(X)$, $\mathcal{H}_\bullet(X)$ and $\mathcal{SH}(X)$ for noetherian schemes and its equivalence to the approach of [MV99] is given in [Rob15], the generalization to arbitrary schemes can be found in [Hoy14, Appendix C].

We follow [Lev18] in writing $X/S \in \mathcal{SH}(S)$ for the $\mathbb{P}^1$–suspension spectrum of a smooth scheme X over S. We will write $X_+ \in \mathcal{H}_\bullet(S)$ for X with a disjoint basepoint added. We sometimes do not distinguish notationally between the pointed motivic space $X_+ \in \mathcal{H}_\bullet(S)$ and its $\mathbb{P}^1$–suspension spectrum $X/S = X_+ \in \mathcal{SH}(S)$.

When dealing with ind-schemes we have elected not to speak of "ind-smooth" schemes and morphisms. Instead, for us a smooth morphisms between ind-schemes will be what is usually called an ind-smooth morphism, namely a formal colimit of smooth morphisms.

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2. Becker–Gottlieb Transfers in Motivic Homotopy Theory

Becker and Gottlieb introduced their eponymous transfer maps in [BG75] as a tool for giving a simple proof of the Adams conjecture. They considered a compact Lie group G and a fiber bundle $E \longrightarrow B$ over a finite CW complex with structure group G and whose fiber F is a closed smooth manifold with a smooth action by G. There is a smooth G-equivariant embedding $F \subset V$ of F into a finite dimensional representation V of G. There is an associated Pontryagin–Thom collapse map $S^V \longrightarrow F^\nu$ where ν is the normal bundle of F in V and F^ν is its Thom space. Denoting by τ the tangent bundle of F one obtains a morphism

$$S^V \longrightarrow F^\nu \longrightarrow F^{\tau \oplus \nu} \simeq F_+ \wedge S^V$$

in G-equivariant homotopy theory. Assuming that $E \longrightarrow B$ is associated to a principal G-bundle $\widetilde{E} \longrightarrow B$ one gets a map

$$\widetilde{E} \times S^V \longrightarrow \widetilde{E} \times (F_+ \wedge S^V)$$

and passing to homotopy orbits with respect to the diagonal G-actions yields the transfer map $B_+ \longrightarrow E_+$ in the stable homotopy category.

This construction of the transfer was generalized in [DP80]. The map $S^V \longrightarrow F_+ \wedge S^V$ arises from a *duality datum* in parameterized stable homotopy theory over the base space B.

Definition 2.1. A *duality datum* in a symmetric monoidal category consists of a pair of objects X and $X^\vee$ with morphisms $\mathbf{1} \xrightarrow{\text{coev}} X \otimes X^\vee$ and $X^\vee \otimes X \xrightarrow{\text{ev}} \mathbf{1}$ such that the compositions

$$X \xrightarrow{\text{coev} \otimes \text{id}} X \otimes X^\vee \otimes X \xrightarrow{\text{id} \otimes \text{ev}} X$$

and

$$X^\vee \xrightarrow{\text{id} \otimes \text{coev}} X^\vee \otimes X \otimes X^\vee \xrightarrow{\text{ev} \otimes \text{id}} X^\vee$$

are identities. In this situation $X^\vee$ is said to be a *right dual* of X and X is said to be a *left dual* of $X^\vee$. If X is additionally a right dual of $X^\vee$, then X is said to be *strongly dualizable* with dual $X^\vee$.

A duality datum in a symmetric monoidal ∞–category $\mathscr{C}$ is a duality datum in the homotopy category $\mathrm{h}\mathscr{C}$, see [Lur12, section 4.6.1].

Remark 2.2. In [Lev18], Levine defines a dual $X^\vee = \mathrm{Map}(X, \mathbf{1})$ for any object X in a *closed* symmetric monoidal category. Then X is called strongly dualizable whenever the induced morphism $X^\vee \otimes X \longrightarrow \mathrm{Map}(X, X)$ is an equivalence. By [Lur12, Lemma 4.6.1.6] this coincides with our definition.

Dold and Puppe show that, for a fiber bundle $E \longrightarrow B$ with fiber a compact smooth manifold, there is a duality datum in the homotopy category of B–parameterized spectra. It exhibits the fiberwise Thom spectrum of the fiberwise stable normal bundle to E as a dual of the suspension spectrum of E. They then show that the transfer in [BG75] is an instance of the following general construction.

Definition 2.3. In a symmetric monoidal ∞–category $\mathscr{C}$, suppose that an object X is equipped with a map $\Delta\colon X \longrightarrow X \otimes C$ for some other object C. Furthermore, suppose that X is strongly dualizable. The *transfer of X with respect to* Δ is defined as the composition

$$\mathrm{tr}_{X,\Delta}\colon \mathbf{1} \xrightarrow{\mathrm{coev}} X \otimes X^{\vee} \xrightarrow{\mathrm{switch}} X^{\vee} \otimes X \xrightarrow{\mathrm{id}\otimes\Delta} X^{\vee} \otimes X \otimes C \xrightarrow{\mathrm{ev}\otimes\mathrm{id}} \mathbf{1} \otimes C \simeq C.$$

If there can be no risk of confusion we write $\mathrm{tr}_X = \mathrm{tr}_{X,\Delta}$.

In Appendix A we construct a symmetric monoidal ∞–category $\mathcal{SH}(B)$ for every smooth ind-scheme B over a base scheme S. This enables us to extend the definition of the motivic Becker–Gottlieb transfer in [Lev18].

Definition 2.4. For a smooth map $f\colon E \longrightarrow B$ between smooth ind-schemes over S with $E/B \in \mathcal{SH}(B)$ strongly dualizable we define the *relative transfer* $\mathrm{Tr}(f/B)\colon \mathbf{1}_B \longrightarrow E/B$ as follows: Applying $f_\#$ to the diagonal $E \longrightarrow E \times_B E$ gives a morphism $\Delta\colon E/B \longrightarrow E/B \wedge E/B$ in $\mathcal{SH}(B)$ and we set $\mathrm{Tr}(E/B) = \mathrm{Tr}(f/B) = \mathrm{tr}_{E/B,\Delta}$.

Additionally, since $\pi\colon B \longrightarrow S$ is a smooth ind-scheme, we can define the *absolute transfer* of f as

$$\mathrm{Tr}(f/S) = \pi_\#(\mathrm{Tr}(f/B))\colon E/S \longrightarrow B/S.$$

Proposition 2.5. *The motivic Becker–Gottlieb transfer enjoys the following properties.*

(i) *The transfer is additive in homotopy pushouts: Suppose X, Y, U and V are smooth ind-schemes over a smooth ind-scheme B over S. Further suppose that there is a homotopy cocartesian square*

$$\begin{array}{ccc} X/B & \longrightarrow & U/B \\ \downarrow & & \downarrow \\ V/B & \longrightarrow & Y/B \end{array}$$

in $\mathcal{SH}(B)$. Assume that Y/B, U/B and V/B are strongly dualizable. Then $\mathrm{Tr}(Y/B)$ is a sum of the compositions

$$\mathbf{1}_B \xrightarrow{\mathrm{Tr}(U/B)} U/B \longrightarrow Y/B$$

$$\mathbf{1}_B \xrightarrow{\mathrm{Tr}(V/B)} V/B \longrightarrow Y/B$$

and

$$\mathbf{1}_B \xrightarrow{\mathrm{Tr}(X/B)} X/B \longrightarrow Y/B$$

in $\mathcal{SH}(B)$.

(ii) *The relative transfer is compatible with pullback: If* $p\colon B' \longrightarrow B$ *and* $f\colon E \longrightarrow B$ *are maps of smooth ind-schemes over* S *and* E/B *is strongly dualizable in* $\mathcal{SH}(B)$ *then the pullback* $p^*(E/B) \simeq (E \times_B B')/B'$ *is strongly dualizable in* $\mathcal{SH}(B')$ *and* $\mathrm{Tr}(p^* f/B') \simeq p^* \mathrm{Tr}(f/B)$.

(iii) *The absolute transfer is natural in cartesian squares: If*

$$\begin{tikzcd} E' \arrow[r] \arrow[d, "f'"'] & E \arrow[d, "f"] \\ B' \arrow[r] & B \end{tikzcd}$$

is a cartesian square of smooth ind-schemes over S *and the vertical maps are smooth, then the square*

$$\begin{tikzcd} E'/S \arrow[r] & E/S \\ B'/S \arrow[r] \arrow[u, "\mathrm{Tr}(f'/S)"] & B/S \arrow[u, "\mathrm{Tr}(f/S)"'] \end{tikzcd}$$

commutes in $\mathcal{SH}(S)$.

To prove part (i) of Proposition 2.5 we appeal to a general additivity result of May's. In the context of symmetric monoidal triangulated categories, [May01] proves that the transfer is additive in distinguished triangles. However, since duality in symmetric monoidal ∞−categories is characterised at the level of homotopy categories, May's theorem admits the following reformulation.

Theorem 2.6 ([May01, Theorem 1.9]). *Let* $\mathcal{C}$ *be a symmetric monoidal stable* ∞*−category and let* $X \longrightarrow Y \longrightarrow Z$ *be a cofiber sequence in* $\mathcal{C}$. *Assume* $C \in \mathcal{C}$ *is such that* $\_ \otimes C$ *preserves cofiber sequences. Suppose that* Y *is equipped with a map* $\Delta_Y\colon Y \longrightarrow Y \otimes C$ *and that* X *and* Y *are strongly dualizable. Then* Z *is strongly dualizable and there are maps* Δ_X *and* Δ_Z *such that*

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\Delta_X"] & Y \arrow[r] \arrow[d, "\Delta_Y"] & Z \arrow[d, "\Delta_Z"] \\ X \otimes C \arrow[r] & Y \otimes C \arrow[r] & Z \otimes C \end{tikzcd}$$

commutes. Furthermore, we have $\mathrm{tr}_{Y,\Delta_Y} = \mathrm{tr}_{X,\Delta_X} + \mathrm{tr}_{Z,\Delta_Z}$ *in* $\pi_0 \operatorname{Map}_{\mathcal{C}}(\mathbf{1}, C)$.

Proof of Proposition 2.5, (i). The homotopy cocartesian square induces a cofiber sequence

$$U/B \vee V/B \longrightarrow Y/B \longrightarrow S^1 \wedge X/B$$

in $\mathcal{SH}(B)$. Shifting this sequence yields and introducing diagonal maps gives a diagram

$$\begin{tikzcd}
X/B \arrow[r] \arrow[d, "\Delta_X"] & U/B \vee V/B \arrow[r] \arrow[d, "\Delta_U \vee \Delta_V"] & Y/B \arrow[dd, "\Delta_Y"] \\
X/B \wedge X/B \arrow[d, "(1)"] & (U/B \wedge U/B) \vee (V/B \wedge V/B) \arrow[d, "(2)"] & \\
X/B \wedge Y/B \arrow[r] & (U/B \vee V/B) \wedge Y/B \arrow[r] & Y/B \wedge Y/B
\end{tikzcd}$$

in which the outer two rows are cofiber sequences and the maps (1) and (2) are induced from the maps $X/B \longrightarrow Y/B$, $U/B \longrightarrow Y/B$ and $V/B \longrightarrow Y/B$ respectively. Then we can conclude using Theorem 2.6. ◻

Part (ii) of Proposition 2.5 is proven in [Lev18, Lemma 1.6]. We formulate the proof of part (iii) as a lemma.

Lemma 2.7. *Let S be a scheme and B and B' smooth ind-schemes over S. Suppose that $f\colon E \longrightarrow B$ is smooth with $E/B \in \mathcal{SH}(B)$ strongly dualizable and*

$$\begin{tikzcd}
E' \arrow[r, "i'"] \arrow[d, "f'"] & E \arrow[d, "f"] \\
B' \arrow[r, "i"'] & B
\end{tikzcd}$$

is cartesian. Then the square

$$\begin{tikzcd}
E'/S \arrow[r, "i'/S"] & E/S \\
B'/S \arrow[r, "i/S"'] \arrow[u, "\mathrm{Tr}(f'/S)"] & B/S \arrow[u, "\mathrm{Tr}(f/S)"']
\end{tikzcd}$$

is homotopy commutative.

Proof. Write $p\colon B' \longrightarrow S$ and $q\colon B \longrightarrow S$ for the structure morphisms. There is a natural transformation $p_\# i^* \longrightarrow q_\#$ defined as the composition

$$p_\# i^* \xrightarrow{\text{unit}} p_\# i^* q^* q_\# \simeq p_\# p^* q_\# \xrightarrow{\text{counit}} q_\#.$$

Consequently, we obtain a homotopy commutative diagram

$$\begin{tikzcd}
B'/S = p_\# p^* \mathbf{1}_S \arrow[r, "p_\# \mathrm{Tr}(f'/B')"] \arrow[d, equal] & p_\# f'_\# i'^* f^* q^* \mathbf{1}_S = E'/S \arrow[d, "\mathrm{Ex}^*_\#"] \\
p_\# i^* q^* \mathbf{1}_S \arrow[r, "p_\# i^* \mathrm{Tr}(f/B)"] \arrow[d] & p_\# i^* f_\# f^* q^* \mathbf{1}_S \arrow[d] \\
B/S = q_\# q^* \mathbf{1}_S \arrow[r, "q_\# \mathrm{Tr}(f/B)"'] & q_\# f_\# f^* q^* \mathbf{1}_S = E/S.
\end{tikzcd}$$

Chasing through the definition of $p_\# i^* \longrightarrow q_\#$ shows that the leftmost composite vertical map is i/S and that the rightmost vertical map is i'/S. $\square$

Finally, we will need some tools to understand when a smooth morphism $E \longrightarrow B$ of smooth ind-schemes over S determines a strongly dualizable object E/B in $\mathcal{SH}(B)$. We have the following formulation of motivic Atiyah duality.

Theorem 2.8 (see [Voe01], [Rio05], [Ayo07b], [CD09]). *If $Y \longrightarrow X$ is a smooth and proper morphism of schemes, then $Y/X \in \mathcal{SH}(X)$ is strongly dualizable.*

Because the property of being strongly dualizable is formulated in the homotopy category, it is immediate that any smooth scheme $Y \longrightarrow X$ such that Y/X is $\mathbb{A}^1$-homotopy equivalent to a smooth and proper scheme over X defines a strongly dualizable object in $\mathcal{SH}(X)$. Furthermore, dualizability is local in the following sense.

Theorem 2.9 ([Lev18, Proposition 1.2, Theorem 1.10]). *Let B be a scheme over S. Suppose that $E \in \mathcal{SH}(B)$ and there is a finite Nisnevich covering family $\{j_i : U_i \longrightarrow B\}$ and that $j_i^* E \in \mathcal{SH}(U_i)$ is strongly dualizable. Then E is strongly dualizable as well.*

If $E \longrightarrow B$ is a Nisnevich–locally trivial fiber bundle with smooth fiber F and B is smooth over S, then E/B is strongly dualizable in $\mathcal{SH}(B)$ if F/S is strongly dualizable in $\mathcal{SH}(S)$

3. Transfers of Grassmannians

Definition 3.1. The ind-scheme Gr_r is the sequential colimit of the Grassmannians $\mathrm{Gr}_r(n)$ of r–planes in n–space along the canonical closed immersions $\mathrm{Gr}_r(n) \hookrightarrow \mathrm{Gr}_r(n+1)$.

It is well known that Gr_r is a model for BGL_r in the $\mathbb{A}^1$–homotopy category. In fact, let $U_r(N)$ be the scheme of monomorphisms $\mathcal{O}^r \longrightarrow \mathcal{O}^N$. Along $\mathcal{O}^N \oplus 0 \subset \mathcal{O}^{N+1}$, there are closed embeddings $U_r(N) \hookrightarrow U_r(N+1)$ and [MV99, Proposition 4.3.7] shows that the colimit $U_r(\infty) = \mathrm{colim}_N\, U_r(N)$ along these embeddings is contractible in $\mathcal{H}(S)$. Also, the quotient $U_r(N)/\mathrm{GL}_r$ is isomorphic to $\mathrm{Gr}_r(N)$ and consequently $U_r(\infty)/\mathrm{GL}_r \cong \mathrm{Gr}_r$ is a model for BGL_r.

Direct sum defines a morphism $U_r(N) \times U_{n-r}(N) \longrightarrow U_n(2N)$ which is equivariant with respect to the block diagonal inclusion $\mathrm{GL}_r \times \mathrm{GL}_{n-r} \longrightarrow \mathrm{GL}_n$. Passing to the colimit $N \to \infty$ and taking quotients yields a morphism

$$i_{r,n} \colon \mathrm{Gr}_r \times \mathrm{Gr}_{n-r} \longrightarrow \mathrm{Gr}_n.$$

This morphism is equivalent in $\mathcal{H}(S)$ to the map $\mathrm{BGL}_r \times \mathrm{BGL}_{n-r} \longrightarrow \mathrm{BGL}_n$ induced by the block diagonal inclusion $\mathrm{GL}_r \times \mathrm{GL}_{n-r} \subset \mathrm{GL}_n$. The goal of this section is to develop a partial inductive description of the absolute transfer $\mathrm{tr}_{n,r} \colon \mathrm{Gr}_{n,+} \longrightarrow \mathrm{Gr}_{r,+} \wedge \mathrm{Gr}_{n-r,+}$ of $i_{r,n}$ in $\mathcal{SH}(S)$. For this purpose a different version of $i_{r,n}$ in $\mathcal{H}(S)$ will be more convenient.

Lemma 3.2. *In $\mathcal{H}(S)$ there is an equivalence $\mathrm{Gr}_r \times \mathrm{Gr}_{n-r} \longrightarrow U_n(\infty)/(\mathrm{GL}_r \times \mathrm{GL}_{n-r})$. Along this equivalence, $i_{r,n}$ corresponds to the quotient*

$$\overline{i_{r,n}}\colon U_n(\infty)/(\mathrm{GL}_r \times \mathrm{GL}_{n-r}) \longrightarrow U_n(\infty)/\mathrm{GL}_n \cong \mathrm{Gr}_n$$

by GL_n.

Proof. Writing $\varphi\colon U_r(N) \times U_{n-r}(N) \longrightarrow U_n(2N)$ for the map induced by taking direct sums, we obtain a commutative diagram

$$\begin{tikzcd}
U_r(N) \times U_{n-r}(N) \arrow[r, "\varphi"] \arrow[d] & U_n(2N) \arrow[d] \\
\mathrm{Gr}_r(N) \times \mathrm{Gr}_{n-r}(N) \arrow[r] \arrow[dr, "i_{r,n}"'] & U_n(2N)/(\mathrm{GL}_r \times \mathrm{GL}_{n-r}) \arrow[d] \\
& \mathrm{Gr}_n(2N).
\end{tikzcd}$$

Passing to the colimit $N \to \infty$ the horizontal maps become equivalences. □

Lemma 3.3. *The morphism $\overline{i_{r,n}}$ is a Zariski–locally trivial bundle over Gr_n. Its fiber is the quotient $\mathrm{GL}_n/(\mathrm{GL}_r \times \mathrm{GL}_{n-r})$.*

Proof. By construction, the morphism $\overline{i_{r,n}}$ is isomorphic to the colimit of the quotient maps $U_n(N)/(\mathrm{GL}_r \times \mathrm{GL}_{n-r}) \longrightarrow U_n(N)/\mathrm{GL}_n \cong \mathrm{Gr}_n(N)$. But these are all Zariski–locally trivial with fiber $\mathrm{GL}_n/(\mathrm{GL}_r \times \mathrm{GL}_{n-r})$. □

We note that $\mathrm{GL}_n/(\mathrm{GL}_r \times \mathrm{GL}_{n-r})$ is equivalent to $\mathrm{Gr}_r(n)$ in $\mathcal{H}(S)$ and this equivalence is compatible with the respective GL_n actions. This is shown in [AHW18, Lemma 3.1.5] and implies in particular that the image in $\mathcal{SH}(\mathrm{Gr}_n)$ of the associated bundle $U_n(\infty) \times^{\mathrm{GL}_n} \mathrm{Gr}_r(n) \longrightarrow \mathrm{Gr}_n$ is equivalent to that of the quotient $U_n(\infty)/(\mathrm{GL}_r \times \mathrm{GL}_{n-r}) \longrightarrow \mathrm{Gr}_n$.

Lemma 3.4. *The morphism $\overline{i_{r,n}}\colon U_n(\infty)/(\mathrm{GL}_r \times \mathrm{GL}_{n-r}) \longrightarrow \mathrm{Gr}_n$ defines a strongly dualizable object $G_{r,n} \in \mathcal{SH}(\mathrm{Gr}_n)$.*

Proof. By Lemma A.3 it will be enough to show that the pullback $i\colon E \longrightarrow \mathrm{Gr}_n(N)$ of $\overline{i_{r,n}}$ along the inclusion $\mathrm{Gr}_n(N) \longrightarrow \mathrm{Gr}_n$ defines a dualizable object in $\mathcal{SH}(\mathrm{Gr}_n(N))$ for all N. But, by Lemma 3.3 the morphism i is a Zariski-locally trivial fiber bundle over $\mathrm{Gr}_n(N)$ with fiber $X = \mathrm{GL}_n/(\mathrm{GL}_r \times \mathrm{GL}_{n-r})$. Hence, to show that i defines a strongly dualizable object in $\mathcal{SH}(\mathrm{Gr}_n(N))$, by Theorem 2.9 it is enough to show that $X/S \in \mathcal{SH}(S)$ is strongly dualizable.

But we have seen that $X \simeq \mathrm{Gr}_r(n)$ in $\mathcal{H}(S)$ and therefore also in $\mathcal{SH}(S)$. The scheme $\mathrm{Gr}_r(n)$ is smooth and proper over S, so motivic Atiyah duality, Theorem 2.8, implies that $\mathrm{Gr}_r(n)/S$ and therefore also X/S is strongly dualizable in $\mathcal{SH}(S)$, see for example [Lev18, Proposition 1.2]. □

Lemma 3.5. *Suppose $r < n$. The open complement of the closed immersion $\mathrm{Gr}_r(n-1) \hookrightarrow \mathrm{Gr}_r(n)$ is the total space of an affine space bundle of rank $n-r$ over $\mathrm{Gr}_{r-1}(n-1)$.*

Dually, the complement of the closed immersion $\mathrm{Gr}_{r-1}(n-1) \hookrightarrow \mathrm{Gr}_r(n)$ is the total space of an affine space bundle of rank r over $\mathrm{Gr}_r(n-1)$.

Proof. Suppose $\mathrm{Spec}(A)$ is an affine scheme mapping to S. On $\mathrm{Spec}(A)$–valued points, the inclusion $\mathrm{Gr}_r(n-1) \hookrightarrow \mathrm{Gr}_r(n)$ is given by considering a projective submodule P of A^{n-1} as a submodule of $A^n = A^{n-1} \oplus A$. It follows that the complement U of $\mathrm{Gr}_r(n-1)$ has $\mathrm{Spec}(A)$–valued points

$$U(\mathrm{Spec}\, A) = \{P \subset A^n : P \text{ is projective of rank } r \text{ and } P \not\subset A^{n-1} \oplus 0\}.$$

Given $P \in U(\mathrm{Spec}\, A)$, the module $P \cap (A^{n-1} \oplus 0)$ will be locally free of rank $r-1$. This gives a map $\varphi\colon U \longrightarrow \mathrm{Gr}_{r-1}(n-1)$ which is trivial over the standard Zariski–open cover of $\mathrm{Gr}_{r-1}(n-1)$ with fiber $\mathbb{A}^{n-r}$.

The dual statement is proved similarly. In fact, the bundle $V \longrightarrow \mathrm{Gr}_r(n-1)$ in question is the tautological r-plane bundle on $\mathrm{Gr}_r(n-1)$. □

The decomposition $\mathrm{Gr}_r(n) = U \cup V$ of the last lemma yields a homotopy cocartesian square

$$\begin{tikzcd}
U \smallsetminus \mathrm{Gr}_{r-1}(n-1) = U \cap V \arrow[r] \arrow[d] & V \simeq \mathrm{Gr}_r(n-1) \arrow[d] \\
\mathrm{Gr}_{r-1}(n-1) \simeq U \arrow[r] & \mathrm{Gr}_r(n)
\end{tikzcd}$$

in the $\mathbb{A}^1$–homotopy category $\mathcal{H}(S)$. It is immediate that this decomposition of $\mathrm{Gr}_r(n)$ is stable under the action of $\mathrm{GL}_{n-1} \times 1 \subset \mathrm{GL}_n$. We can therefore pass to the bundles over Gr_{n-1} associated to the universal GL_{n-1}–torsor $U_{n-1}(\infty)$ over Gr_{n-1} and obtain a homotopy cocartesian square

$$\begin{tikzcd}
(U_{n-1}(\infty) \times^{\mathrm{GL}_{n-1}} (U \cap V))/\mathrm{Gr}_{n-1} \arrow[r] \arrow[d] & G_{r,n-1} \arrow[d] \\
G_{r-1,n-1} \arrow[r] & (U_{n-1}(\infty) \times^{\mathrm{GL}_{n-1}} \mathrm{Gr}_r(n))/\mathrm{Gr}_{n-1}
\end{tikzcd}$$

in $\mathcal{SH}(\mathrm{Gr}_{n-1})$.

Proposition 3.6. *Suppose $r < n$ and consider the composition*

$$\varphi\colon \mathrm{Gr}_{n-1,+} \xrightarrow{\mathrm{incl}} \mathrm{Gr}_{n+} \xrightarrow{\mathrm{tr}_{n,r}} \mathrm{Gr}_{r+} \wedge \mathrm{Gr}_{n-r,+}$$

where incl *is given by the assignment $P \longmapsto P \oplus A$ on* $\mathrm{Spec}(A)$*–valued points. Then there is a map $\psi\colon \mathrm{Gr}_{n-1,+} \longrightarrow \mathrm{Gr}_{r-1,+} \wedge \mathrm{Gr}_{n-r,+}$ in $\mathcal{SH}(S)$ such that φ is the sum of the compositions*

$$\mathrm{Gr}_{n-1,+} \xrightarrow{\mathrm{tr}_{n-1,r}} \mathrm{Gr}_{r,+} \wedge \mathrm{Gr}_{n-1-r,+} \xrightarrow{\mathrm{id} \wedge \mathrm{incl}} \mathrm{Gr}_{r,+} \wedge \mathrm{Gr}_{n-r,+}$$
$$\mathrm{Gr}_{n-1,+} \xrightarrow{\mathrm{tr}_{n-1,r-1}} \mathrm{Gr}_{r-1,+} \wedge \mathrm{Gr}_{n-r,+} \xrightarrow{\mathrm{incl} \wedge \mathrm{id}} \mathrm{Gr}_{r,+} \wedge \mathrm{Gr}_{n-r,+}$$

and

$$\mathrm{Gr}_{r-1,+} \xrightarrow{\psi} \mathrm{Gr}_{r-1,+} \wedge \mathrm{Gr}_{n-r,+} \xrightarrow{\mathrm{incl} \wedge \mathrm{id}} \mathrm{Gr}_{r,+} \wedge \mathrm{Gr}_{n-r,+}.$$

Proof. Consider the homotopy pullback

$$\begin{tikzcd} E = U_{n-1}(\infty) \times^{\mathrm{GL}_{n-1}} \mathrm{Gr}_r(n) \arrow[r] \arrow[d] & \mathrm{Gr}_r \times \mathrm{Gr}_{n-r} \arrow[d] \\ \mathrm{Gr}_{n-1} \arrow[r, "\mathrm{incl}"'] & \mathrm{Gr}_n \end{tikzcd}$$

in $\mathscr{H}(S)$. By the discussion following Lemma 3.5 we obtain a cofiber sequence

$$X/\mathrm{Gr}_{n-1} \longrightarrow G_{r,n-1} \vee G_{r-1,n-1} \longrightarrow E/\mathrm{Gr}_{n-1}$$

in $\mathcal{SH}(\mathrm{Gr}_{n-1})$ where $X = U_{n-1}(\infty) \times^{\mathrm{GL}_{n-1}} (U \cap V)$. Theorem 2.6 then shows that

$$\mathrm{tr}_{E/\mathrm{Gr}_{n-1}} = \mathrm{tr}_{G_{r,n-1}} + \mathrm{tr}_{G_{r-1,n-1}} - \mathrm{tr}_{X/\mathrm{Gr}_{n-1}}$$

in $\mathcal{SH}(\mathrm{Gr}_{n-1})$. Passing to the absolute transfer and using Lemma 2.7 yields that φ is the sum of the compositions

$$\mathrm{Gr}_{n-1,+} \xrightarrow{\mathrm{tr}_{n-1,r}} \mathrm{Gr}_{r,+} \wedge \mathrm{Gr}_{n-1-r,+} \xrightarrow{\mathrm{id} \wedge \mathrm{incl}} \mathrm{Gr}_{r,+} \wedge \mathrm{Gr}_{n-r,+}$$

$$\mathrm{Gr}_{n-1,+} \xrightarrow{\mathrm{tr}_{n-1,r-1}} \mathrm{Gr}_{r-1,+} \wedge \mathrm{Gr}_{n-r,+} \xrightarrow{\mathrm{incl} \wedge \mathrm{id}} \mathrm{Gr}_{r,+} \wedge \mathrm{Gr}_{n-r,+}$$

and

$$\mathrm{Gr}_{n-1,+} \longrightarrow X_+ \longrightarrow \mathrm{Gr}_{r,+} \wedge \mathrm{Gr}_{n-r,+}$$

in $\mathcal{SH}(S)$. Here, the map $X_+ \longrightarrow \mathrm{Gr}_{r,+} \wedge \mathrm{Gr}_{n-r,+}$ is obtained from the inclusion $U \cap V \subset \mathrm{Gr}_r(n)$ by passing to associated bundles. Now, this inclusion factors through the inclusion of U into $\mathrm{Gr}_r(n)$. By Lemma 3.5 the inclusion $\mathrm{Gr}_{r-1}(n-1) \subset U$ is an $\mathbb{A}^1$–equivalence, being the zero section of an affine space bundle. Therefore $X_+ \longrightarrow \mathrm{Gr}_{r,+} \wedge \mathrm{Gr}_{n-r,+}$ factors through the map $\mathrm{incl} \wedge \mathrm{id}\colon \mathrm{Gr}_{r-1,+} \wedge \mathrm{Gr}_{n-r,+} \longrightarrow \mathrm{Gr}_{r,+} \wedge \mathrm{Gr}_{n-r,+}$. This way we obtain the map ψ and the required decomposition of $\mathrm{tr}_{n,r} \circ \mathrm{incl}$. □

4. Proof of the Theorem

We have the filtration

$$\mathrm{Gr}_{0,+} \xrightarrow{i_1} \mathrm{Gr}_{1,+} \xrightarrow{i_2} \dots \xrightarrow{i_n} \mathrm{Gr}_{n,+} \longrightarrow \dots \xrightarrow{i_m} \mathrm{Gr}_{m,+}$$

and we have seen that for $r \leq n$ the map $i_{r,n}\colon \mathrm{Gr}_r \times \mathrm{Gr}_{n-r} \longrightarrow \mathrm{Gr}_n$ admits an absolute transfer $\mathrm{tr}_{n,r}\colon \mathrm{Gr}_{n,+} \longrightarrow \mathrm{Gr}_{r,+} \wedge \mathrm{Gr}_{n-r,+}$ in the motivic stable homotopy category $\mathcal{SH}(S)$. Write $f_{n,r}\colon \mathrm{Gr}_{n,+} \longrightarrow \mathrm{Gr}_{r,+}$ for the composition

$$\mathrm{Gr}_{n,+} \xrightarrow{\mathrm{tr}_{n,r}} \mathrm{Gr}_{r,+} \wedge \mathrm{Gr}_{n-r,+} \xrightarrow{\mathrm{proj}} \mathrm{Gr}_{r,+}$$

and $\phi_{n,r}$ for the composition

$$\mathrm{Gr}_{n,+} \xrightarrow{f_{n,r}} \mathrm{Gr}_{r,+} \longrightarrow \mathrm{Gr}_r/\mathrm{Gr}_{r-1}.$$

LEMMA 4.1. *With notation as above, for $r < n$ the compositions*

$$\mathrm{Gr}_{n-1,+} \xrightarrow{i_n} \mathrm{Gr}_{n,+} \xrightarrow{f_{n,r}} \mathrm{Gr}_{r,+} \longrightarrow \mathrm{Gr}_r/\mathrm{Gr}_{r-1}$$

and

$$\mathrm{Gr}_{n-1,+} \xrightarrow{f_{n-1,r}} \mathrm{Gr}_{r,+} \longrightarrow \mathrm{Gr}_r/\mathrm{Gr}_{r-1}$$

coincide.

Proof. By Proposition 3.6 the composition $f_{n,r} \circ i_n$ is a sum of two compositions

$$\mathrm{Gr}_{n-1,+} \xrightarrow{\mathrm{tr}_{n-1,r}} \mathrm{Gr}_{r,+} \wedge \mathrm{Gr}_{n-1-r,+} \xrightarrow{\mathrm{id}\wedge\mathrm{incl}} \mathrm{Gr}_{r,+} \wedge \mathrm{Gr}_{n-r,+} \xrightarrow{\mathrm{proj}} \mathrm{Gr}_{r,+}$$

and

$$\mathrm{Gr}_{n-1,+} \longrightarrow \mathrm{Gr}_{r-1,+} \wedge \mathrm{Gr}_{n-r,+} \xrightarrow{\mathrm{incl}\wedge\mathrm{id}} \mathrm{Gr}_{r,+} \wedge \mathrm{Gr}_{n-r,+} \xrightarrow{\mathrm{proj}} \mathrm{Gr}_{r,+}$$

in $\mathcal{SH}(S)$. But the composition

$$\mathrm{Gr}_{r-1,+} \xrightarrow{\mathrm{incl}} \mathrm{Gr}_{r,+} \longrightarrow \mathrm{Gr}_r/\mathrm{Gr}_{r-1}$$

vanishes. Therefore, $f_{n,r} \circ i_n$ coincides with the composition

$$\mathrm{Gr}_{n-1,+} \xrightarrow{f_{n-1,r}} \mathrm{Gr}_{r,+} \longrightarrow \mathrm{Gr}_r/\mathrm{Gr}_{r-1}$$

in $\mathcal{SH}(S)$. □

Proof of Theorem 1.1. Proceeding by induction on n, assume that

$$\Phi = \bigvee_{r=0}^{n-1} \phi_{n-1,r} \colon \mathrm{Gr}_{n-1,+} \longrightarrow \bigvee_{r=0}^{n-1} \mathrm{Gr}_r/\mathrm{Gr}_{r-1}$$

is an equivalence in $\mathcal{SH}(S)$. Because of Lemma 4.1 we have a commutative diagram

$$\begin{tikzcd} & \mathrm{Gr}_{n,+} \arrow[dr, "\Phi'"] & \\ \mathrm{Gr}_{n-1,+} \arrow[ur, "i_n"] \arrow[rr, "\Phi"'] & & \bigvee_{r=0}^{n-1} \mathrm{Gr}_r/\mathrm{Gr}_{r-1} \end{tikzcd}$$

where $\Phi' = \bigvee_{r=0}^{n-1} \phi_{n,r}$. It follows that $\Phi^{-1} \circ \Phi' \circ i_n \simeq \mathrm{id}$, i. e. i_n admits a left inverse. That is to say, the cofiber sequence

$$\mathrm{Gr}_{n-1,+} \xrightarrow{i_n} \mathrm{Gr}_{n,+} \longrightarrow \mathrm{Gr}_n/\mathrm{Gr}_{n-1}$$

splits and yields an equivalence

$$\mathrm{Gr}_{n,+} \xrightarrow{(\Phi^{-1}\Phi')\vee\phi_{n,n}} \mathrm{Gr}_{n-1,+} \vee \mathrm{Gr}_n/\mathrm{Gr}_{n-1}$$

since $\phi_{n,n}$ is by definition the canonical projection. Post-composing with $\Phi \vee \mathrm{id}$ then shows that the stable map $\Phi' \vee \phi_{n,n} \colon \mathrm{Gr}_{n,+} \longrightarrow \bigvee_{r=0}^{n} \mathrm{Gr}_r/\mathrm{Gr}_{r-1}$ is an equivalence in $\mathcal{SH}(S)$ as well. $\square$

A. Stable Motivic Homotopy Theory of Smooth Ind-Schemes

We freely use the theory of presentable ∞–categories as developed in [Lur09, section 5.5.3]. The ∞–category of presentable ∞–categories with left adjoints as morphisms is denoted $\mathcal{P}r^{\mathrm{L}}$ while the ∞–category of presentable ∞–categories with right adjoints as morphisms is denoted $\mathcal{P}r^{\mathrm{R}}$. There is an equivalence $\mathcal{P}r^{\mathrm{L}} \simeq (\mathcal{P}r^{\mathrm{R}})^{\mathrm{op}}$ of ∞–categories which is the identity on objects and sends a left adjoint functor to its right adjoint. Both $\mathcal{P}r^{\mathrm{L}}$ and $\mathcal{P}r^{\mathrm{R}}$ are complete and cocomplete and the homotopy limits in both $\mathcal{P}r^{\mathrm{L}}$ and $\mathcal{P}r^{\mathrm{R}}$ coincide with homotopy limits in the ∞–category of ∞–categories.

Definition A.1. A *smooth ind-scheme* over S is an object of $\mathrm{Ind}(\mathrm{Sm}_S)$, the ∞–category of ind-objects in the category of smooth schemes over S with arbitrary morphisms between them. A morphism of ind-schemes is smooth if it can be presented as a colimit of smooth morphisms in Sm_S.

The goal of this section will be to generalize the definition of the stable motivic homotopy category $\mathcal{SH}$ to smooth ind-schemes over S. Our approach is to use part of the six functor formalism for $\mathcal{SH}$, as established in [Ayo07b; Ayo07a] for noetherian schemes and extended to arbitrary schemes in [Hoy14, Appendix C]. An overview of the standard functorialities, at least at the level of triangulated categories, can be found in [CD09].

The first functoriality of $\mathcal{SH}$ can be summarized as follows. For every morphism $f \colon X \longrightarrow Y$ between smooth schemes over S we have an adjunction

$$f^* : \mathcal{SH}(X) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{SH}(Y) : f_*$$

between the stable presentable ∞–categories $\mathcal{SH}(X)$ and $\mathcal{SH}(Y)$. These adjunctions assemble into functors $\mathcal{SH}^* \colon \mathrm{Sm}_S^{\mathrm{op}} \longrightarrow \mathcal{P}r^{\mathrm{L}}$ and $\mathcal{SH}_* \colon \mathrm{Sm}_S \longrightarrow \mathcal{P}r^{\mathrm{R}}$ which are naturally equivalent after composing with the equivalence $\mathcal{P}r^{\mathrm{L}} \simeq (\mathcal{P}r^{\mathrm{R}})^{\mathrm{op}}$. If $f \colon X \longrightarrow Y$ is smooth, then there is an additional adjunction

$$f_\# : \mathcal{SH}(Y) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{SH}(X) : f^*.$$

These assemble into a functor $\mathcal{SH}_\#\colon \mathrm{Sm}_{S,\mathrm{sm}} \longrightarrow \mathcal{P}r^{\mathrm{L}}$ from the wide subcategory of Sm_S consisting of smooth morphisms between smooth schemes over S. There are various exchange transformations associated with a cartesian square

$$\begin{tikzcd} \bullet \arrow[r, "g"] \arrow[d, "q"'] & \bullet \arrow[d, "p"] \\ \bullet \arrow[r, "f"'] & \bullet \end{tikzcd}$$

in Sm_S, of which we only mention the transformation

$$\mathrm{Ex}^*_\#\colon g_\# q^* \longrightarrow p^* f_\#$$

when f and hence g is smooth. More details on these exchange transformations may be found in [CD09].

Because $\mathcal{P}r^{\mathrm{R}}$ is cocomplete, the functor $\mathcal{SH}_*$ naturally extends to a functor

$$\mathcal{SH}_*\colon \mathrm{Ind}(\mathrm{Sm}_S) \longrightarrow \mathcal{P}r^{\mathrm{R}}$$

and we obtain a functor

$$\mathcal{SH}^*\colon \mathrm{Ind}(\mathrm{Sm}_S)^{\mathrm{op}} \longrightarrow \mathcal{P}r^{\mathrm{L}}$$

by again composing with the equivalence $\mathcal{P}r^{\mathrm{L}} \simeq (\mathcal{P}r^{\mathrm{R}})^{\mathrm{op}}$.

More explicitly, if $(X_i)_{i\in I}$ is a filtered diagram of smooth schemes over S and $X = \mathrm{colim}_i X_i$ as an ind-scheme over S, then

$$\mathcal{SH}^*(X) = \operatorname*{holim}_i \mathcal{SH}^*(X_i) \quad \text{and} \quad \mathcal{SH}_*(X) = \operatorname*{hocolim}_i \mathcal{SH}_*(X_i).$$

Note that $\mathcal{SH}^*(X)$ and $\mathcal{SH}_*(X)$ are equivalent ∞–categories since homotopy limits along left adjoints in $\mathcal{P}r^{\mathrm{L}}$ correspond to homotopy colimits along their right adjoints in $\mathcal{P}r^{\mathrm{R}}$, see [Lur09, section 5.5.3]. This description of $\mathcal{SH}(X)$ also shows that it inherits the structure of a closed symmetric monoidal, stable, presentable ∞–category, see [Lur12, section 3.4.3, Proposition 4.8.2.18].

The adjunction $f^* \dashv f_*$ for a morphism $f\colon X \longrightarrow Y$ of ind-schemes is obtained by presenting f as a colimit of maps $f_i\colon X_i \longrightarrow Y_i$ between schemes over S and then taking f^* to be the functor induced on the homotopy limits in $\mathcal{P}r^{\mathrm{L}}$ and f_* the functor induced on the homotopy colimits in $\mathcal{P}r^{\mathrm{R}}$.

It remains to construct the extra left-adjoint $f_\#$ for a smooth map f between ind-schemes over S. First, a morphism $f\colon X \longrightarrow Y$ between ind-schemes is smooth if and only if it is a filtered colimit of smooth maps $f_i\colon X_i \longrightarrow Y_i$. Each f_i^* admits a left adjoint $f_{i\#}$ and since $\mathcal{P}r^{\mathrm{R}}$ is stable under limits, the functor $f^*\colon \mathcal{SH}^*(Y) \longrightarrow \mathcal{SH}^*(X)$ admits a left adjoint as well. That is to say, $\mathcal{SH}^*\colon \mathrm{Ind}(\mathrm{Sm}_S)^{\mathrm{op}} \longrightarrow \mathcal{P}r^{\mathrm{L}}$ restricts to a functor $\mathcal{SH}^*\colon \mathrm{Ind}(\mathrm{Sm}_S)^{\mathrm{op}}_{\mathrm{sm}} \longrightarrow \mathcal{P}r^{\mathrm{R}}$ from the wide subcategory of $\mathrm{Ind}(\mathrm{Sm}_S)$ consisting of smooth maps between smooth ind-schemes over S. Composing with the equivalence $\mathcal{P}r^{\mathrm{L}} \simeq (\mathcal{P}r^{\mathrm{R}})^{\mathrm{op}}$ then yields the functor

$$\mathcal{SH}_{\#} \colon \mathrm{Ind}(\mathrm{Sm}_S)_{\mathrm{sm}} \longrightarrow \mathcal{P}r^{\mathrm{L}}.$$

In summary, we have the following proposition.

Proposition A.2. *For every ind-scheme X over S, there is a closed symmetric monoidal, stable, presentable ∞−category $\mathcal{SH}(X)$. For every morphism $f\colon X \longrightarrow Y$ between ind-schemes there is an associated adjunction*

$$f^* : \mathcal{SH}(Y) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{SH}(X) : f_*$$

with f^* a monoidal functor. If f is smooth then there is an additional adjunction

$$f_\# : \mathcal{SH}(X) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{SH}(Y) : f^*.$$

These data are functorial in f and admit various natural exchange transformations. If X happens to be a smooth scheme over S then this version of $\mathcal{SH}(X)$ is naturally equivalent to the usual construction.

Following [Lev18], for a smooth morphism $f\colon X \longrightarrow Y$ of ind-schemes over S we define $X/Y = f_\#(\mathbf{1}_X) \in \mathcal{SH}(Y)$ where $\mathbf{1}_X$ denotes the monoidal unit in $\mathcal{SH}(X)$. In particular, if $Y = S$, we see that any smooth ind-scheme X over S determines an object $X/S \in \mathcal{SH}(S)$. If X is a smooth scheme over S, then X/S is canonically equivalent to the $\mathbb{P}^1$−suspension spectrum of X in $\mathcal{SH}(S)$; see [Ayo14, Lemma C.2].

Lemma A.3. *Suppose B is a smooth ind-scheme over S and $E \in \mathcal{SH}(B)$. If B is presented as a filtered colimit $B = \mathrm{colim}_i\, B_i$ of smooth schemes in $\mathrm{Ind}(\mathrm{Sm}_S)$, let $f_i\colon B_i \longrightarrow B$ be the canonical map for each i. Then $E \in \mathcal{SH}(B)$ is strongly dualizable if and only if $f_i^*E \in \mathcal{SH}(B_i)$ is strongly dualizable for every i.*

Proof. This follows from [Lur12, Proposition 4.6.1.11] since we have $\mathcal{SH}(B) \simeq \lim_i \mathcal{SH}(B_i)$. □

Proposition A.4. *Suppose an ind-scheme X is presented as a colimit $X = \mathrm{colim}_i\, X_i$ in $\mathrm{Ind}(\mathrm{Sm}_S)$. Then there is a natural equivalence $X/S \simeq \mathrm{hocolim}_i\, X_i/S$ in $\mathcal{SH}(S)$.*

Proof. Write $\pi\colon X \longrightarrow S$ and $\pi_i\colon X_i \longrightarrow S$ for the structure morphisms. Suppose $Y \in \mathcal{SH}(S)$ is arbitrary. Then we have natural equivalences

$$\begin{aligned}
\mathrm{Map}_{\mathcal{SH}(S)}(\pi_\#\mathbf{1}_X, Y) &\simeq \mathrm{Map}_{\mathcal{SH}(X)}(\mathbf{1}_X, \pi^*Y)\\
&\simeq \operatorname*{holim}_i \mathrm{Map}_{\mathcal{SH}(X_i)}(\mathbf{1}_{X_i}, \pi_i^*Y)\\
&\simeq \operatorname*{holim}_i \mathrm{Map}_{\mathcal{SH}(S)}(\pi_{i\#}\mathbf{1}_{X_i}, Y)\\
&\simeq \mathrm{Map}_{\mathcal{SH}(S)}(\operatorname*{hocolim}_i X_i/S, Y)
\end{aligned}$$

of mapping spaces. The Yoneda lemma implies that $X/S = \pi_\#\mathbf{1}_X \simeq \mathrm{hocolim}_i\, X_i/S$ in $\mathcal{SH}(S)$. □

This proposition allows us to extend the definition of the functor $\_/S\colon \mathrm{Sm}_S \longrightarrow \mathcal{SH}(S)$ in [Lev18] to ind-schemes. The functor $\_/S\colon \mathrm{Sm}_S \longrightarrow \mathcal{SH}(S)$ extends uniquely up to natural equivalence to a functor $\_/S\colon \mathrm{Ind}(\mathrm{Sm}_S) \longrightarrow \mathcal{SH}(S)$ because $\mathcal{SH}(S)$ is cocomplete. By Proposition A.4 this coincides on objects with the previous construction $\pi_\#(\mathbf{1}_X)$ for a smooth ind-scheme $\pi\colon X \longrightarrow S$.

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